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Covariant Hamiltonian Formalisms for Particles and Antiparticles

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The hyperplane and proper time formalisms are discussed mainly for spin-half particles in the quantum case. A connection between these covariant Hamiltonian formalisms is established. It is shown that by choosing the spacelike hyperplanes instantaneously orthogonal to the direction of motion of the particle allows one to retrieve the proper time formalism on the mass shell. As a consequence, the relation between the Stückelberg–Feynman picture and the standard canonical picture of quantum field theory is clarified.

1. INTRODUCTION

The unification of the principles of relativity and quantum mechanics presents a serious obstacle. On the one hand, as from the seminal work of Minkowski (1908), the first theory deals with space and time on an equal footing: "Space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality."

On the other hand, the principles of quantum mechanics, originally developed in a canonical formalism, have broken this symmetry by choosing the coordinate time of a given frame of reference to label the evolution of the system. Therefore the standard canonical formalism does not provide a relativistic invariant description of the dynamical evolution of the system. Moreover, in this framework it is not possible to describe simultaneously particles and antiparticles at a first-quantized level.

Different covariant formalisms have been proposed to overcome these obstacles. Significant advances were obtained when the problem of reformu-

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lating the old canonical theory of quantum electrodynamics was addressed. Two kinds of solutions were presented at the Pocono conference half a century ago (Schweber, 1986; Mehra, 1994), one by Schwinger (1948), who, like Tomonaga (1946), essentially generalized the standard canonical formalism of quantum field theory to arbitrary spacelike surfaces, and another, containing more radical changes, by Feynman (1951) (see also Schweber, 1986; Mehra, 1994). Feynman's ideas, like Stückelberg's (1941a, b, 1942), dealt with the space-time trajectories of charged particles, and were essentially formulated at a first-quantized level.

In this work we recall such ideas in order to discuss the covariant Hamiltonian formalism for relativistic particles. We shall focus on the Fleming hyperplane formalism (Fleming, 1965, 1966), closely related to the Tomonaga–Schwinger ideas,[2] and the Feynman proper time formalism, with the aim of establishing a bridge between them. This will clarify many misunderstood issues of the connection between the standard canonical picture and the Feynman space-time picture, from the Feynman point of view.[3]

The main purpose of this work is to discuss Feynman's formalism for relativistic spin-1/2 particles in the quantum case (Sections 4 and 5), but for pedagogical reasons we begin by discussing the problem at the classical level for the spinless relativistic particle in Sections 2 and 3.

Throughout this work we use natural units ($\hbar = c = 1$). Our convention for the metric is

$$ds^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu, \qquad \eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1) \tag{1}$$

2. THE CLASSICAL RELATIVISTIC PARTICLE IN THE HYPERPLANE

The standard canonical formalism usually considers particle states only. Let us go beyond this formalism by using the Hamiltonian

$$H = \varepsilon\sqrt{m^2 + \mathbf{p}^2} \tag{2}$$

where ε is $+1$ and -1 for particles and antiparticles, respectively. In this way we adopt Stückelberg's (1941a, b, 1942) and Feynman's (1948, 1949) ideas introducing the concept of antiparticles at the classical level as negative energy states going backward in coordinate time.[4] As we stressed above, the canonical formalism privileges the temporal coordinate x^0 of a reference

[2] See Jauch and Rohrlich (1976) for a list of references about formalisms involving arbitrary spacelike surfaces.
[3] From the canonical point of view, this connection was established by Dyson (1948).
[4] Such a notion can be also used for deriving the Dirac equation from first principles (Gaioli and Garcia Alvarez, 1995).

frame in order to describe the evolution, the one in which H is the temporal component of the four-momenta $p_\mu = (H, -\mathbf{p})$. In other words, for each reference frame we have a different Hamiltonian which generates the dynamical evolution of the system in the corresponding coordinate time. The key idea of the hyperplane formalism is to unify such a multiplicity of dynamical descriptions by taking the temporal coordinate of a privileged frame, τ, for labeling the dynamics. This choice can be written in an invariant language as follows: Space-time is foliated with a family of spacelike surfaces $n^\mu x_\mu - \tau = 0$ (the hyperplanes) characterized by a normalized vector n^μ ($n^\mu n_\mu = 1$) orthogonal to the surfaces

$$n^\mu = \frac{\partial x^\mu}{\partial \tau} \tag{3}$$

We have chosen the normal vector n^μ in the direction of the future light cone. Following this convention, the components of the temporal vector n^μ take the simple form $(n_\tau)^\mu = (1, 0, 0, 0)$ in the coordinates $(x_\tau)^\mu = (\tau, \mathbf{x}_\tau)$ of the privileged frame. Of course, the choice of such a privileged foliation, in order to label the dynamics, is arbitrary. Usually, each observer adopts his canonical foliation, the one corresponding to coordinate time. But at this point the hyperplane formalism dissociates the dual (geometrical and dynamical) role of the temporal coordinates of the different reference systems. Each temporal coordinate retains its geometrical role, but only one (arbitrarily chosen) adopts the dynamical one. Note that in this sense the time τ registered by the privileged coordinate frame is an absolute scalar parameter. That is, any lapse of $\tau(P_1) - \tau(P_2)$ corresponding to two events at points P_1 and P_2 in space-time is (by definition) independent of the coordinate systems chosen.[5] As a consequence, its conjugate variable, the Hamiltonian $H(n)$, is also scalar, which becomes evident writing it as[6]

$$H(n) = n^\mu p_\mu \tag{4}$$

In this way the hyperplane formalism describes the evolution of the system from any coordinate system in a covariant way. However, note that the multiplicity of dynamical descriptions of the standard canonical formalism discussed above was not lost. It is hidden in the arbitrary choice of n^μ. The only thing that has been improved is that now the canonical formalism is independent of the coordinate system chosen. That is, the canonical formalism provides a relativistic invariant description of the dynamical evolution of the system.

[5] What is relative is the lapse in the time cooordinates, i.e., for two systems S and S', $x^0(P_1) - x^0(P_2) \neq x^0(P_1) - x^0(P_2)$.

[6] Notice also that particle and antiparticle concepts are interchanged if we reverse the direction of n^μ.

Although the expression (4) looks explicitly covariant, the canonical formalism is rather complicated because the variables p_μ are not independent, since they satisfy the mass-shell constraint

$$p^\mu p_\mu = m^2 \tag{5}$$

An alternative expression for (4) can be obtained in terms of the canonical momentum $\mathbf{p}_\tau$ conjugate to the hyperplane coordinates $\mathbf{x}_\tau$. Using (2) in coordinates $(\tau, \mathbf{x}_\tau)$, we have

$$H(n) = \varepsilon\sqrt{m^2 + p^\mu(n)p_\mu(n)} \tag{6}$$

where[7]

$$p^\mu(n) = p^\mu - n^\mu(n^\nu p_\nu) \tag{7}$$

is the four-vector associated with $\mathbf{p}_\tau$ [which in the coordinates of the privileged frame reads $(p_\tau)^\mu(n) = (0, \mathbf{p}_\tau)$]. However, the new momentum variables do not simplify the problem, because they also satisfy a constraint

$$p^\mu(n)n_\mu = 0 \tag{8}$$

since $p^\mu(n)$ is the projection of p^μ to the hyperplane $\tau = 0$.

Notice also that the covariant Poisson brackets,

$$\{f, g\}_{xp} \equiv \frac{\partial f}{\partial x^\alpha}\frac{\partial g}{\partial p_\alpha} - \frac{\partial g}{\partial x^\alpha}\frac{\partial f}{\partial p_\alpha}$$

of (7) with the four-vector

$$x^\mu(n) = x^\mu - n^\mu(n^\nu x_\nu) \tag{9}$$

associated with $\mathbf{x}_\tau$, is not canonical

$$\{x^\mu(n), p^\nu(n)\}_{xp} = \eta^{\mu\nu} - n^\mu n^\nu \tag{10}$$

We will return to this kind of problem later.

Up to this point, we have not removed the undesirable arbitrariness in the choice of n^μ. In the case of the free particle the only four-vector that gives a privileged direction in space-time is the four-velocity (which for a spinless particle also coincides with the direction of its four-momentum[8]). We can remove the arbitrariness by choosing

$$n^\mu = \varepsilon\frac{dx^\mu}{ds} = \varepsilon\frac{p^\mu}{m} \tag{11}$$

[7] The expression (6) was generalized to a curved space-time by Ferraro *et al.* (1987).
[8] This is not the case in general when the particle has spin. See, for example, Corben (1961, 1968).

which identifies the canonical variable τ with the proper time of the particle,

$$\tau = \varepsilon s\,, \qquad ds = \varepsilon\sqrt{1 - \mathbf{v}^2}\, dx^0 \tag{12}$$

since this choice imposes our privileged system to be a system in which the particle is at rest.[9] Using (11) and the constraint (5) in (4), we have

$$H(n) = \varepsilon m = \varepsilon\sqrt{p^\mu p_\mu} \tag{13}$$

which shows that for particles the conjugate variable of τ becomes the rest mass m.

3. THE PROPER TIME FORMALISM FOR CLASSICAL SPINLESS PARTICLES

Recently Hall and Anderson (1995) proposed a covariant Hamiltonian formalism for a relativistic particle based on a square-root super-Hamiltonian

$$\mathcal{H} = \sqrt{p^\mu p_\mu} \tag{14}$$

which resembles expression (13), but with the four-momentum p^μ not restricted to the mass shell ($\mathcal{H}$ behaves as a positive variable mass). Such a formalism originally developed by Moses (1969) and Johnson (1969) and more recently discussed by Evans (1990), Hannibal (1991a), and us (Aparicio *et al.* 1995a, b) is a formalism free from constraints in which the invariant evolution parameter is identified with the proper time. In this framework, in contrast with (10), we have covariant commutation relations

$$\{x^\mu, p^\nu\}_{xp} = \eta^{\mu\nu} \tag{15}$$

However, we must deal with an indefinite-mass system[10] in such a way that the standard notion of definite-mass particles and antiparticles are recovered by specifying the initial conditions.

Hall and Anderson's approach is interesting because, in spite of postulating the form of the Hamiltonian, they derive $\mathcal{H}$ in a constructive way by imposing physical requirements.

Their argument flows as follows: Let us assume a four-dimensional Hamiltonian formalism whose equations of motion read

$$\frac{dx^\mu}{d\lambda} = \frac{\partial\mathcal{H}}{\partial p_\mu}, \qquad \frac{dp^\mu}{d\lambda} = -\frac{\partial\mathcal{H}}{\partial x_\mu} \tag{16}$$

where λ is an invariant evolution parameter and $\mathcal{H} = \mathcal{H}(x, p)$ is the covariant

[9] Note that s is the proper time for both particles and antiparticles, since, according to Stückelberg, for antiparticles $dx^0 < 0$.

[10] Notice that (15) is incompatible with the mass-shell constraint (5).

Hamiltonian. Let us assume that the invariant evolution parameter can be identified with the proper time,

$$\lambda = s \tag{17}$$

Such a condition, in principle, allows us univocally to determine the form of the free spinless Hamiltonian. In this case space-time homogeneity imposes that $\mathcal{H}$ cannot explicitly depend on x^μ, so $\mathcal{H} = \mathcal{H}(p^\mu)$, and the condition of being a scalar under Lorentz transformations leads $\mathcal{H}$ to be an arbitrary function of $p = \sqrt{\eta_{\mu\nu}p^\mu p^\nu}$, the only scalar that we can form with the available tensors of the theory. Using the equations of motion, we can write the constraint

$$\eta_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 1 \tag{18}$$

as

$$\eta_{\mu\nu}\frac{\partial \mathcal{H}}{\partial p_\mu}\frac{\partial \mathcal{H}}{\partial p_\nu} = 1 \tag{19}$$

or, taking into account that $\partial\mathcal{H}/\partial p_\mu = (d\mathcal{H}/dp)\, p^\mu/p$, as

$$\left(\frac{d\mathcal{H}}{dp}\right)^2 = 1 \tag{20}$$

Finally, the differential equation (20) can be easily integrated to give

$$\mathcal{H} = \pm p \tag{21}$$

which is the Moses–Johnson Hamiltonian. The four-velocity associated with this Hamiltonian is

$$\frac{dx^\mu}{ds} = \pm\frac{p^\mu}{p} \tag{22}$$

an equation which shows that for positive mass states we have $\lambda = s$, with the sign specified in equation (12).

Hall and Anderson have also generalized this argument for the case in which the theory admits another four-vector t^μ, giving in this case a Hamiltonian of the type[11]

$$\mathcal{H} = t^\mu p_\mu \tag{23}$$

[11] Note that the new Hamiltonian is a particular case of the Hamiltonian (23) for $t^\mu = \pm p^\mu/p$.

This $\mathscr{H}$ is admissible provided that the norm of t^{μ} satisfies

$$t^{\mu} t_{\mu} = 1 \tag{24}$$

The equation of motion of the four-velocity is

$$\frac{dx^{\mu}}{d\lambda} = t^{\mu} \tag{25}$$

so the constraint (18) is equivalent to condition (24). We also remark the analogy of Hamiltonians (4) and (23). In the conclusion of their work Hall and Anderson speculate on the possibility of incorporating spin from such a generalization. At the end of this work we show that this conjecture can be crystallized by relaxing the normalization condition (24), by choosing t^{μ} as the Dirac matrix γ^{μ}.

4. THE DIRAC EQUATION IN THE HYPERPLANE FORMALISM

In this section we review the hyperplane formalism for a quantum spinning particle described by the Dirac equation,

$$\gamma^{\nu} i\partial_{\nu}\psi(x) = m\psi(x) \tag{26}$$

(Hammer *et al.*, 1968), with the aim of establishing a connection with the proper time formalism in an analogous way to the one discussed at the end of Section 2. Then our purpose is to translate the Hamiltonian form of equation (26),

$$i\partial_0\psi(x) = (\boldsymbol{\alpha} \cdot \mathbf{p} + \beta m)\psi(x) \tag{27}$$

($p_{\mu} = i\partial_{\mu}$) into an arbitrary hyperplane.

There are two ways for doing this, depending on whether we take equation (26) or equation (27) as a starting point. We begin by discussing the first possibility, proposed by Czachor (1995), which is more straightforward. Multiplying both members of equation (26) by γ^{ν} and taking into account the identity $\gamma^{\mu}\gamma^{\nu} = \eta^{\mu\nu} - i\sigma^{\mu\nu}$, we have (Kálnay and MacCotrina, 1968)

$$i\partial^{\mu}\psi(x) = (i\sigma^{\mu\nu}p_{\nu} + m\gamma^{\mu})\psi(x) \tag{28}$$

Now contracting equation (28) with n^{ν}, we finally obtain (Czachor, 1995)

$$i\frac{\partial\psi(x)}{\partial\tau} = n_{\mu}(i\sigma^{\mu\nu}p_{\nu} + m\gamma^{\mu})\psi(x) \tag{29}$$

where we have used the chain rule and (3)

$$\frac{\partial\psi}{\partial\tau} = \frac{\partial\psi}{\partial x^{\mu}}\frac{\partial x^{\mu}}{\partial\tau} = n^{\mu}\partial_{\mu}\psi$$

The original derivation of Hammer *et al.* (1968) follows a similar argument to the one used for obtaining expression (6). It departs from equation (27) in the privileged reference system and rewrites this equation in a covariant way. Adapted to our notation and conventions, the Hammer–MacDonald–Pursey equation reads

$$i\frac{\partial\psi}{\partial\tau} = H(n)\psi \tag{30}$$

$$H(n) = [\alpha^{\mu}(n)p_{\mu}(n) + \beta(n)m]$$

where $\alpha^{\mu}(n)$ and $\beta(n)$ are the four-vector and scalar matrixes associated with the Dirac matrixes α^{i} and β in the privileged frame, namely

$$\alpha^{\mu}(n) = i\sigma^{\mu\nu}n_{\nu} \tag{31}$$

$$\beta(n) = n_{\mu}\gamma^{\mu}$$

[This can be easily checked by remembering that $(n_{\tau})^{\mu} = (1,0,0,0)$].

The parameter τ is, in general, unrelated to the proper time but, as discussed above, classically one can always choose the coordinate system in such a way that εn_{μ} can be identified with the velocity of the particle $dx^{\mu}/ds = \pi^{\mu}/m$ ($\pi^{\mu} = p^{\mu} - eA^{\mu}$). Then $\varepsilon\tau$ coincides with s. The same identification cannot be made at the quantum level in general, because the concept of trajectory is lost. However, in the free case we can argue that this identification makes sense for the eingenstates of the momentum operator

$$p^{\mu}\psi_{k}(x) = k^{\mu}\psi_{k}(x) \tag{32}$$

At least in this case, the second member of the second equality in equation (11) is well defined.[12] By choosing $n_{\mu} = \varepsilon k_{\mu}/m$, the first term in the second member of equation (29) vanishes, and we finally obtain

$$i\frac{\partial\psi_{k}(x)}{\partial s} = p_{\mu}\gamma^{\mu}\psi_{k}(x)\,, \qquad s = \varepsilon\tau \tag{33}$$

Equation (33) resembles the Feynman parametrization of the Dirac equation (Feynman, 1951). However, note that the whole formalism discussed here is restricted to the mass shell ($k^{\mu}k_{\mu} = m^{2}$), since $\psi_{k}(x)$ satisfies the Dirac

[12] However, note that this is not the case in the presence of interactions because we have no chance to have a common basis which diagonalizes the four-vector operator π^{μ} since $[\pi_{\mu},\pi_{\nu}] = -ieF_{\mu\nu}$.

equation (26). In the next section we briefly discuss the formalism associated with an equation like (33) out of the mass shell.

5. THE PROPER TIME PARAMETRIZATION OF THE DIRAC EQUATION

Feynman in 1948, in his dissertation at the Pocono conference (Feynman, 1951; Schweber, 1986; Mehra, 1994), introduced a fifth parameter in the Dirac equation[13]

$$-i\frac{\partial\Psi(x, s)}{\partial s} = \mathcal{H}\Psi(x, s) \tag{34}$$

$$\mathcal{H} = p_\mu\gamma^\mu$$

to formulate a manifestly covariant (multiple-time) formalism of quantum electrodynamics.

Equation (34) is a Schrödinger-like equation in which the scalar Hamiltonian $\mathcal{H}$ plays the role of a mass operator. Notice that we retrieve the Dirac equation as an eigenvalue equation, $\mathcal{H}\Psi_m = m\Psi_m$, for stationary states $\Psi_m(x, s) = \Psi_m(x, 0)e^{ims}$. The evolution operator

$$U = e^{ip_\mu\gamma^\mu s} \tag{35}$$

is unitary in the indefinite scalar product

$$\langle\Psi, \Phi\rangle = \int\overline{\Psi}(x)\Phi(x)\, d^4x \tag{36}$$

The "norm" is positive and negative for particle and antiparticle states, respectively (Gaioli and Garcia Alvarez, 1993). Moreover, such indefiniteness has its root in the Stückelberg picture, i.e., it can be shown that at the semiclassical level (Gaioli and Garcia Alvarez, 1996)

$$\text{sign}\,[\overline{\Psi}(x, s)\,\Psi(x, s)] = \text{sign}\frac{dx^0}{ds} \tag{37}$$

The evolution of any operator A in the Heisenberg picture is given by

$$\frac{dA}{ds} = -i[\mathcal{H}, \mathrm{A}] \tag{38}$$

which is the proper time derivative originally proposed by Beck (1942).

[13] The difference between the sign of equation (33) and the sign of equation (34) is a consequence of having considered different starting points. While equation (33) is a direct covariant generalization of equation (27) in the hyperplane formalism, equation (34) is motivated by an off-shell proper time formalism, which for the spatial components preserves the standard results (Aparicio *et al.*, 1995a).

During the last 50 years, this kind of parametrization and the proper time derivative have been rediscovered or discussed by many authors from different motivations (Nambu, 1950; Enatsu, 1954; Davidon, 1955; Proca, 1954, 1955; Gürsey, 1957; Peres and Rosen, 1960; Szamosi, 1961, 1963; Rafanelli, 1967a, b, 1968, 1970; DeVos and Hilgevoord, 1967; Bunge and Kálnay, 1969; Kálnay and MacCotrina, 1969; Johnson, 1971; Johnson and Chang, 1974; López and Pérez, 1981; Herdergen, 1982; Kubo, 1985; Sherry, 1989; Hannibal, 1991a, 1994; Grossmann and Peres, 1963; Schwinger, 1975; Rumpf, 1979; Barut, 1988; Barut and Thacker, 1985; Barut and Pavsik, 1987; Evans, 1990; Fanchi, 1993a, b; Czachor and Kuna, 1997).

In previous work (Aparicio *et al.*, 1995a) we established the connection between the derivative (38) and other proper time derivatives discussed in the literature.[14] We have concluded that this is the most satisfactory approach for incorporating the notion of proper time into the Dirac theory at the quantum level. In other works we have discussed the interpretation of the formalism (Aparicio *et al.*, 1995b; Gaioli and Garcia Alvarez) and the de Sitter invariance of equation (34) (Garcia Alvarez and Gaioli, 1997a, b). For the sake of completeness, we review in this section some points necessary to understand the material discussed in the previous ones.

We begin by noticing that in equation (34) the coordinate time x^0 has been elevated to the status of an operator, canonically conjugate to the energy p^0. Their commutation relation and the standard canonical one for the three-position and momentum can be summarized in the covariant commutation relation

$$[x^\mu, p^\nu] = -i\eta^{\mu\nu} \tag{39}$$

which is the quantum analogue of equation (15). It is possible because, as in the formalism of Section 3, the mass-shell constraint (5) satisfied by the irreducible representations of the Poincaré group is no longer valid. In this case, there is a new dynamical group of symmetries that enlarges the Poincaré group, that is, the de Sitter group, which could have been taken as the starting point to obtain the Feynman parametrization (Garcia Alvarez and Gaioli, 1997a, b). Here we have followed the heuristic argument given in Section 4 to obtain the form of a covariant Hamiltonian conjugate to the proper time s on the mass shell and then we extrapolated this form out of the mass shell. We conclude this section by giving an independent argument which shows that the operator $p_\mu\gamma^\mu$ determines the evolution of the system in the proper time s.

[14] See Fanchi (1993b) for a review of different proposals.

Using (38), for $A = x^\mu$, and (39) we obtain the covariant generalization of Breit's (1928, 1931) formula

$$\frac{dx^\mu}{ds} = \gamma^\mu \tag{40}$$

Projecting this equation of motion on positive and negative mass states, for eliminating the covariant *Zitterbewegung*, by means of the projectors

$$\Lambda_\pm \equiv \frac{1}{2}\left(1 \pm \frac{\mathcal{H}}{\sqrt{\mathcal{H}^2}}\right) \tag{41}$$

$$\Lambda_\pm \mathcal{H} \Lambda_\pm = \pm\Lambda_\pm \sqrt{p^\mu p_\mu}\Lambda_\pm \tag{42}$$

we have

$$\Lambda_\pm \frac{dx^\mu}{ds}\Lambda_\pm = \pm\Lambda_\pm \frac{p^\mu}{\sqrt{p^\mu p_\mu}}\Lambda_\pm \tag{43}$$

The projected Hamiltonian and four-velocity are analogues of (21) and (22), respectively, which on the positive mass shell leads us to the identification of the evolution parameter with the proper time. Moreover, we see that eliminating the interference between positive and negative states, we have the analogue of the proper time constraint (18), namely

$$\Lambda_\pm \frac{dx^\mu}{ds}\Lambda_\pm \Lambda_\pm \frac{dx_\mu}{ds}\Lambda_\pm = \Lambda_\pm \tag{44}$$

6. FURTHER REMARKS AND CONCLUSIONS

Summarizing, the standard canonical formalism has two difficulties:

(*a*) It does not provide a relativistic invariant description of the dynamical evolution of the system.

(*b*) It does not enable us to include simultaneously particles and antiparticle states.

The problem (*a*) arises because the coordinate time is not a Lorentz scalar, and (*b*) is due to the fact that particles and antiparticles go forward and backward in this time, respectively. Then the coordinate time is not able to describe processes involving both species simultaneously. The standard canonical formalism of quantum field theory is a many-particle formalism with negative and positive charges for the particles and antiparticles, respectively.[15] Such a picture reinterprets the notion of antiparticle of the Stückelberg

[15] This double sign of the charge has its correlate in the double sign of the kinetic energy, $dx^0/ds = \varepsilon\sqrt{m^2 + p^2}/m$, in the Feynman–Stückelberg picture, while the sign of the energy and the charge is kept unaltered in the standard picture and in the Stückelberg one, respectively.

picture by reversing the direction of its space-time trajectory, which is equivalent to taking the conjugate of its charge.[16]

The first difficulty (*a*) is removed by the Fleming formalism, but there is a price to be paid:

(*a*′) It has an arbitrariness in the choice of the privileged system.

We have shown that as soon as we try to remove this arbitrariness by choosing $n^\mu = \varepsilon dx^\mu/ds$, we get the proper time formalism on the mass shell. But in this case difficulty (*b*) remains. We have to label the dynamics with the time $s = \varepsilon\tau$ (in this case τ is the proper time of the particle) to have the same label for both particles and antiparticles, a solution which naturally arises in the proper time formalism out of the mass shell.

The last discussion suggests to us how to remove difficulty (*b*) at the level of the hyperplane formalism. One could label the dynamics with $\xi = \varepsilon\tau$. In this case the Hamiltonian corresponding to equation (4) and to the Dirac equation in the hyperplane, reads

$$H_\xi(n) = \varepsilon n^\mu p_\mu \tag{45}$$

$$i\frac{\partial\psi(x)}{\partial\xi} = \varepsilon n_\mu(i\sigma^{\mu\nu}p_\nu + m\gamma^\mu)\psi(x) \tag{46}$$

Note that the new Hamiltonian, like the rest mass, becomes positive definite. The hyperplane formalism corresponding to equation (45) out of the mass shell is equivalent to the Hall and Anderson formalism, identifying t^μ with εn^μ. Moreover, it is interesting to see the analogy between (45) with the covariant Hamiltonian (34), identifying εn^μ (whose temporal component is $n^0_\tau = \varepsilon$) with γ^μ (notice that the eingenvalues of γ^0 are ± 1).

Finally, in contrast with the standard case, the scalar product associated with the new Dirac equation (46) is indefinite, i.e.,

$$\langle\psi, \psi\rangle = \int \bar{\psi}\gamma_\mu\psi\, d\sigma^\mu_\xi = \varepsilon \int \bar{\psi}\gamma_\mu\psi n^\mu\, d\sigma_\tau \tag{47}$$

As in the case of equation (36), this indefiniteness arises as a consequence of describing particle and antiparticle dynamics with the same label.

To summarize, the relation between the standard canonical picture and the Feynman–Stückelberg one can be synthesized as follows:

In the first one the mass, the kinetic energy, and the scalar product are always positive definite. Both particles and antiparticles go forward in coordinate time and proper time, and they are only distinguished by the sign of the charge. In the second case, both particles and antiparticles have positive

[16] This property, emphasized by Feynman (1948, 1949) at the classical level, also holds in the quantum case (Garcia Alvarez and Gaioli, 1997b).

mass, but only the proper time evolution goes forward for both species. Particles and antiparticles have positive and negative kinetic energy, respectively, and according to this they go forward and backward in coordinate time. The charges are the same for both species, which avoids the use of a many-particle formalism in order to describe particle creation and annihilation processes.[17] As a consequence we also have an indefinite scalar product, something which goes against our standard notions. Moreover, this is why Dirac disregarded the Klein–Gordon equation (Weinberg, 1995). But, like the double sign in the energy, it has its roots in the indefinite metric of the Minkowski space-time manifold (1).[18] In other words, while the second picture seems to be a natural way for adapting the principles of quantum mechanics to the theory of relativity, the first one looks like a deliberate attempt to keep our old picture of nonrelativistic quantum mechanics for describing the full relativistic quantum phenomena.

REFERENCES

Aparicio, J. P., Gaioli, F. H., and Garcia Alvarez, E. T. (1995a). *Physical Review A*, **51**, 96.

Aparicio, J. P., Gaioli, F. H., and Garcia Alvarez, E. T. (1995b). *Physics Letters A*, **200**, 233.

Barut, A. O. (1988). *Foundations of Physics*, **18**, 95.

Barut, A. O., and Pavsic, M. (1987). *Classical and Quantum Gravity*, **4**, L41.

Barut, A. O., and Thacker, W. (1985). *Physical Review D*, **31**, 1186.

Beck, G. (1942). *Revista Faculdade de Ciencias de Coimbra*, **10**, 66.

Bleuler, K. (1950). *Helvetica Physica Acta*, **23**, 567.

Breit, G. (1928). *Proceedings of the National Academy of Science USA*, **14**, 553.

Breit, G. (1931). *Physical Review*, **37**, 90.

Bunge, M., and Kálnay, A. J. (1969). *Progress of Theoretical Physics*, **42**, 1445.

Cohen-Tannoudji, C., Dupont-Roc, J., and Grynberg, G. (1989). *Photons and Atoms: Introduction to Quantum Electrodynamics*, Wiley, New York.

Corben, H. C. (1961). *Physical Review*, **121**, 1833.

Corben, H. C., (1968). *Classical and Quantum Theories of Spinning Particles*, Holden-Day, San Fancisco.

Czachor, M. (1995). Personal communication.

Czachor, M., and Kuna M. (1997). *Off-shell indefinite-metric triple-bracket generalization of the Dirac equation*, in *Proceedings of the XXI International Colloquium on Group Theoretical Methods in Physics (Group 21)*, H. E. Doebner, W. Scherer, and P. Natermann, eds., World Scientific, Singapore, Vol. 1, p. 451.

Davidon, W. C. (1955). *Physical Review*, **97**, 1131.

DeVos, J. A., and Hilgevoord, J. (1967). *Nuclear Physics B*, **1**, 494.

Dyson, F. J. (1949). *Physical Review*, **75**, 486.

[17] Remember that, at the semiclassical level, such processes can be pictured as a zigzag trajectory in space-time (Feynman, 1948, 1949; Garcia Alvarez and Gaioli, 1997).

[18] Notice that an indefinite metric space also appears in the covariant quantization of electromagnetic fields (Gupta, 1950; Bleuler, 1950; see also Jauch and Rohrlich, 1976, Chapter 6, and Cohen Tannoudji *et al.*, 1989, Chapter 5).

Enatsu, H. (1954). *Progress of Theoretical Physics*, **11**, 125.
Enatsu, H., and Kawaguchi, S. (1975). *Nuovo Cimento A*, **27**, 468.
Evans, A. B. (1990). *Foundations of Physics*, **20**, 309.
Fanchi, J. R. (1993a). *Parametrized Relativistic Quantum Theory*, Kluwer, Boston.
Fanchi, J. R. (1993b). *Foundations of Physics*, **23**, 487.
Ferraro, R., Yastremiz, C., and Castagnino, M. (1987). *Physical Review D*, **35**, 540.
Feynman, R. P. (1948). *Physical Review*, **74**, 939.
Feynman, R. P. (1949). *Physical Review*, **76**, 749.
Feynman, R. P. (1951). *Physical Review*, **84**, 108.
Fleming, G. N. (1965). *Physical Review*, **137B**, 188.
Fleming, G. N. (1966). *Journal of Mathematical Physics*, **7**, 1959.
Gaioli F. H., and Garcia Alvarez, E. T. (1993). *Anales AFA*, **5**, 39.
Gaioli F. H., and Garcia Alvarez, E. T. (1995). *American Journal of Physics*, **63**, 177.
Gaioli, F. H., and Garcia Alvarez, E. T. (1996). Proper time approach to the localization problem II. The covariant Zitterbewegung, IAFE preprint.
Garcia Alvarez, E. T., and Gaioli, F. H. (1997a). The role of the de Sitter group in relativistic quantum mechanics, in *Proceedings of the XXI International Colloquium on Group Theoretical Methods in Physics (Group 21)*, H. E. Doebner, W. Scherer, and P. Natermann eds., World Scientific, Singapore, Vol. 1, p. 514.
Garcia Alvarez, E. T., and Gaioli, F. H. (1997b). *International Journal of Theoretical Physics*, **36**, 2391.
Grossmann, Z., and Peres, A. (1963). *Physical Review*, **132**, 2346.
Gupta, S. N. (1950). *Proceeding of the Physical Society of London A*, **63**, 681.
Gürsey, F. (1957). *Nuovo Cimento*, **5**, 786.
Hall M. J. W., and Anderson M. R. (1995). *American Journal of Physics*, **63**, 633.
Hammer, C. L., MacDonald, S. C., and Pursey, P. L. (1968). *Physical Review*, **171**, 1349.
Hannibal, L. (1991a). *International Journal of Theoretical Physics*, **30**, 1431.
Hannibal, L. (1991b). *International Journal of Theoretical Physics*, **30**, 1445.
Hannibal, L. (1994). *Foundations of Physics Letters*, **7**, 551.
Herdegen, A. (1982). *Acta Physica Polonica B*, **13**, 863.
Jauch, J. M., and Röhrlich, F. (1976). *The Theory of Photons and Electrons*, Springer, Berlin, p. 9.
Johnson, J. E. (1969). *Physical Review*, **181**, 1755.
Johnson, J. E. (1971). *Physical Review D*, **3**, 1735.
Johnson, J. E., and Chang, K. K. (1974). *Physical Review D*, **10**, 2421.
Kálnay, A. J., and MacCotrina, E. (1969). *Progress of Theoretical Physics*, **42**, 1422.
Kubo, R. (1985). *Nuovo Cimento A*, **35**, 293.
Lopez, C. A., and Perez, A. M. (1981). *Nuovo Cimento Letters*, **30**, 173.
Mehra, J. (1994). *The Beat of a Different Drum. The Life and Science of Richard Feynman*, Clarendon, Oxford, Chap. 15.
Minkowski, H. (1908). *Space and Time*, reprinted in *The principle of Relativity*, Dover, New York (1952), p. 75.
Moses, H. E. (1969). *Annals of Physics*, **52**, 444.
Nambu, Y. (1950). *Progress of Theoretical Physics*, **5**, 82.
Peres, A., and Rosen, N. (1960). *Nuovo Cimento*, **18**, 644.
Proca, A. (1954). *Le Journal de Physique et le Radium*, **15**, 65.
Proca, A. (1955). *Nuovo Cimento*, **2**, 962.
Rafanelli, K. (1967a). *Nuovo Cimento A*, **52**, 342.
Rafanelli, K. (1967b). *Journal of Mathematical Physics*, **8**, 1440.
Rafanelli, K. (1968). *Physical Review*, **175**, 1761.

Rafanelli, K. (1970). *Nuovo Cimento A*, **67**, 48.
Rumpf, H. (1979). *General Relativity and Gravitation*, **10**, 509.
Schweber, S. S. (1986). *Review of Modern Physics*, **58**, 449.
Schwinger, J. (1948). *Physical Review*, **10**, 1439.
Schwinger, J. (1975). *Physical Review D*, **12**, 3105.
Sherry, G. C. (1989). *Foundations of Physics*, **19**, 733.
Stückelberg, E. C. G. (1941a). *Helvetica Physica Acta*, **14**, 322.
Stückelberg, E. C. G. (1941b). *Helvetica Physica Acta*, **14**, 558.
Stückelberg, E. C. G. (1942). *Helvetica Physica Acta*, **15**, 23.
Szamosi, G. (1961). *Nuovo Cimento*, **20**, 1090.
Szamosi, G. (1963). *Nuovo Cimento*, **29**, 677.
Tomonaga, S. (1946). *Progress of Theoretical Physics*, **1**, 27.
Weinberg, S. (1995). *The Quantum Theory of Fields*, Cambridge University Press, Cambridge, p. 7.